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Coherent stochastic resonance in the presence of a field

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A recent paper by Bulsara, Lowen, and Rees [Phys. Rev. E **49**, 4989 (1994)] presents a perturbation analysis of coherent stochastic resonance in a half-space in the presence of a field. We believe that the analysis there was flawed due to an improper use of the method of images and that a correct version of a perturbation analysis can be given by using a transformation of the underlying equations. The result still exhibits stochastic resonance.

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I. INTRODUCTION

The theory and applications of the set of phenomena classified under the rubric of stochastic resonance (SR) has recently been reviewed by Moss [1] and Jung [2]. To date there have been a number of physical systems in which some form of SR can be demonstrated [3–5] as well as neurological processes, which are conjectured to make use of SR [6–8]. There are many theoretical formulations of SR, the simplest of which is the one-dimensional overdamped dynamical system defined by the equation

$$\dot{y} = -F(y) + b\cos(\nu t) + f(t) \qquad (1)$$

in which $F(y)$ is a deterministic force, b is a constant amplitude, ν is the frequency of the periodic forcing term, and $f(t)$ is some form of noise. Stochastic resonance arises as a type of interaction between the periodic forcing term and the noise. The characteristic of SR is that the sinusoidal driving force is able to significantly enhance or in some way qualitatively change effects on the dynamics that are due to noise alone.

In addition to the periodic and random forces, one also needs a nonlinear mechanism to produce SR since in a linear system there is essentially no interaction between random and deterministic signals. In the absence of noise, a weak (in comparison to a potential barrier assumed as part of the definition of the dynamic system) periodic force is unable to induce transitions between different potential minima. When all three elements are present in the system the periodic forcing term can produce SR when the frequency resonates with the Kramers jump rate. An earlier attempt to show that SR can occur without a periodic signal [9] was found to be in error [10].

There are, however, linear systems that exhibit SR due to the presence of traps [11]. Although the system remains linear, the trapping boundaries produce additional stable or metastable states. The possibility of SR in terms of the mean first-passage time (MFPT) to trapping is due to this factor. This was shown to be true for a one-dimensional lattice random walk and a pure diffusion process on a line terminated by two traps [11]. The MFPT to trapping in such a system was shown to have a minimum when considered as a function of either the frequency or the amplitude of the periodic field [12]. The motivation for studying the behavior of such systems is as a model for systems in which elements of a system subject to periodic fields can change behavior when a threshold is reached. These are exemplified by excitable cells in which the amplitude of nerve signals that reach a threshold are returned to their resting state [13] and in the study of induced transitions between the two states of a Schmitt trigger [14].

In the present paper we consider the resonance phenomenon in a one-dimensional diffusion process on a semi-infinite line driven by a periodic forcing term and a constant bias. The bias term is necessitated by the fact that the MFPT of a diffusion process on a semi-infinite line terminated by a trapping point is infinite. A perturbation analysis of such a a system was given in [15], but we believe that analysis was flawed because of an inappropriate use of the method of images to derive a solution to a diffusion equation in which the position of the trapping point is time dependent. It is known that the existence of moving boundaries greatly increases the complexity of finding a solution [16,17] and, in particular, that such a solution cannot be found using the method of images.

II. ANALYSIS

When the noise term in Eq. (1) is such that $\langle f(t)\rangle = 0$ and $\langle f(t)f(t')\rangle = 2D\delta(t-t')$, D having the dimensions of a diffusion constant, then the properties of the random variable $y(t)$ may be summarized in terms of a probability density at time t, which we write as $p(y,t)$. The Fokker-Planck equation satisfied by this function can be written

$$\frac{\partial p}{\partial t}=D\frac{\partial^2 p}{\partial y^2}-[\mu+b\cos(\nu t)]\frac{\partial p}{\partial y} , \tag{2}$$

where, in the case to be considered here, μ is taken to be a positive constant and the trap is located at $y=a$. Equation (2) can be written in terms of dimensionless parameters by changing the spatial variable to $x=1-y/a$, the time variable to $\tau=Dt/a^2$, the amplitude of the constant force to $v=\mu a/D$, the amplitude of the periodic force to $\varepsilon=ba/D$, and the frequency to $\omega=\nu a^2/D$. In this set of coordinates the trap is located at $x=0$ and the initial position is located at $x=1$, which means that x can take on all positive values. This set of transformations converts Eq. (2) to

$$\frac{\partial p}{\partial \tau}=\frac{\partial^2 p}{\partial x^2}+[v+\varepsilon\cos(\omega\tau)]\frac{\partial p}{\partial x} . \tag{3}$$

Our perturbation theory is based on the assumptions that $v=O(1)$ and $\varepsilon\ll 1$.

It will prove convenient to eliminate the term in v by defining a dependent variable $\Gamma(x,\tau)$ by

$$p(x,\tau)=\Gamma(x,\tau)\exp\left[-\frac{vx}{2}-\frac{v^2\tau}{4}\right] . \tag{4}$$

This way of eliminating the constant forcing term has the advantage of keeping the trap at the fixed point $x=0$ rather than having a moving trap as in [15]. The function $\Gamma(x,\tau)$ introduced in Eq. (4) satisfies

$$\frac{\partial \Gamma}{\partial \tau}=\frac{\partial^2 \Gamma}{\partial x^2}+\varepsilon\left[\frac{\partial \Gamma}{\partial x}-\frac{v}{2}\Gamma\right]\cos(\omega\tau) , \tag{5}$$

as can be verified by substitution into Eq. (3) for $p(x,\tau)$.

Since ε is assumed to be a small parameter we will expand the function Γ in a perturbation series as

$$\Gamma(x,\tau)=\sum_{n=0}^{\infty}\Gamma_n(x,\tau)\varepsilon^n , \tag{6}$$

concentrating only on the two lowest-order terms in the expansion. These are readily seen to satisfy

$$\begin{aligned}\frac{\partial \Gamma_0}{\partial \tau}&=\frac{\partial^2 \Gamma_0}{\partial x^2} ,\\ \frac{\partial \Gamma_1}{\partial \tau}-\frac{\partial^2 \Gamma_1}{\partial x^2}&=\left[\frac{\partial \Gamma_0}{\partial x}-\frac{v}{2}\Gamma_0\right]\cos(\omega\tau)\\ &\equiv F(x,\tau)\cos(\omega\tau) .\end{aligned} \tag{7}$$

The initial conditions for these equations are

$$\Gamma_0(x,0)=\delta(x-1) , \quad \Gamma_1(x,0)=0 . \tag{8}$$

An elementary calculation serves to show that $\Gamma_0(x,\tau)$ is

$$\Gamma_0(x,\tau)=\frac{1}{\sqrt{4\pi\tau}}\{e^{-(x-1)^2/4\tau}-e^{-(x+1)^2/4\tau}\}e^{v/2} , \tag{9}$$

with the result that the function $F(x,\tau)$ that appears in Eq. (7) is

$$F(x,\tau)=\frac{1}{4\sqrt{\pi\tau^3}}\{(x+1+v\tau)e^{-(x+1)^2/4\tau}-(x-1+v\tau)e^{-(x-1)^2/4\tau}\}e^{v/2} . \tag{10}$$

A formal solution to the equation for $\Gamma_1(x,\tau)$ can be written in terms of the Green's function for the diffusion equation, which incorporates the trap at $x=0$. This is found to be

$$\Gamma_1(x,\tau)=\frac{1}{\sqrt{4\pi}}\int_0^\tau \frac{\cos(\omega\tau')}{\sqrt{\tau-\tau'}}d\tau'\int_0^\infty F(u,\tau')\{e^{-(x-u)^2/4(\tau-\tau')}-e^{-(x+u)^2/4(\tau-\tau')}\}du . \tag{11}$$

The integral with respect to u can be evaluated explicitly. If we denote the integrand by $J(x,u,\tau')$, then it is found that

$$J(x,u,\tau')=\frac{(u+v\tau')e^{v/2}}{2\sqrt{\pi(\tau')^3}}\left\{\exp\left[-\frac{1}{4\tau}(x+1)^2-\frac{\tau}{4\tau'(\tau-\tau')}\left[u-1+(x+1)\frac{\tau'}{\tau}\right]^2\right]-\exp\left[-\frac{1}{4\tau}(x-1)^2-\frac{\tau}{4\tau'(\tau-\tau')}\left[u-1+(x-1)\frac{\tau'}{\tau}\right]^2\right]\right\} . \tag{12}$$

Since this has a relatively simple dependence on u the integral can be evaluated in closed form, the results being expressible in terms of the functions

$$f(b,\lambda)=\left[\frac{\pi}{4\lambda}\right]^{1/2}\operatorname{erf}(-b\sqrt{\lambda}) , \quad g(b,\lambda)=bf(b,\lambda)+\frac{1}{2\lambda}\exp(-\lambda b^2) , \quad h(\tau,x)=\exp\left\{-\frac{1}{4\tau}(1+x)^2\right\} . \tag{13}$$

Let $I(\tau,\tau';x)=\int_0^\infty J(x,u,\tau')du$. This function is found to be

$$I(\tau,\tau';x)=\frac{1}{\sqrt{4\pi(\tau')^3}}\left\{v\tau'\left[h(\tau,x)f\left[1-(x+1)\frac{\tau'}{\tau}\right]-h(\tau,-x)f\left[1+(x-1)\frac{\tau'}{\tau}\right]\right]\right\}+\frac{1}{\sqrt{4\pi(\tau')^3}}\left[h(\tau,x)g\left[1-(x+1)\frac{\tau'}{\tau}\right]-h(\tau,-x)g\left[1+(x-1)\frac{\tau'}{\tau}\right]\right] , \tag{14}$$

which then leaves only the integral with respect to τ to be evaluated in the expression for $\Gamma_1(x,t)$. In the following discussion we show that it is possible to find an explicit formula for the MFPT.

The simplest parameter that exhibits SR is the MFPT corresponding to the time to reach the trap. We sketch a derivation of the expression for this parameter. A representation of this function can be obtained by making use of Eq. (4) as

$$\langle t(\omega)\rangle=\int_0^\infty d\tau\int_0^\infty p(x,\tau)dx=\hat{\Gamma}\left[\frac{v}{2},\frac{v^2}{4}\right], \quad (15)$$

in which $\hat{\Gamma}(s,s')$ is the two-dimensional Laplace transform of $\Gamma(x,\tau)$ with respect to its two arguments. Let us decompose $\langle t(\omega)\rangle$ into a sum of two terms, the first being the contribution from $\Gamma_0(x,\tau)$, which, of course, cannot give rise to resonant behavior, and the second contribution coming from $\Gamma_1(x,\tau)$, which is proportional to ε. We omit this factor in calculating the Laplace transform to concentrate on the contribution from $\Gamma_1(x,\tau)$ only.

For this purpose we begin by defining the function

$$G(x,u,\tau)=\frac{e^{v/2}}{\sqrt{4\pi\tau}}\{e^{-(x-u)^2/4\tau}-e^{-(x+u)^2/4\tau}\} \quad (16)$$

and seeing, from Eq. (11), that $\Gamma_1(x,\tau)$ can be expressed in the form

$$\Gamma_1(x,\tau)=\int_0^\infty du\int_0^\tau F(u,\tau')\cos(\omega\tau')G(x,u,\tau-\tau')d\tau' . \quad (17)$$

The transform of this function with respect to τ is simplified by noting that the τ' integral is in convolution form. This implies that

$$\mathcal{L}_\tau\{\Gamma_1(x,\tau)\}=\tfrac{1}{2}\int_0^\infty\left\{\bar{F}\left[u,\frac{v^2}{4}+i\omega\right]+\bar{F}\left[u,\frac{v^2}{4}-i\omega\right]\right\}\times\bar{G}\left[x,u,\frac{v^2}{4}\right]du , \quad (18)$$

in which, according to Eq. (15), the transform parameter is set equal $v^2/4$ and the overbar indicates a transform with respect to τ. The final step is to take the transform with respect to x. Define the notation $\mathcal{L}_x\{\bar{G}(x,u,s)\}=\tilde{G}(s',u,s)$. Then our final result can be written in terms of the two-dimensional Laplace transform as

$$\langle t(\omega)\rangle_1=\tfrac{1}{2}\int_0^\infty\left\{\bar{F}\left[u,\frac{v^2}{4}+i\omega\right]+\bar{F}\left[u,\frac{v^2}{4}-i\omega\right]\right\}\times\tilde{G}\left[\frac{v}{2},u,\frac{v^2}{4}\right]du . \quad (19)$$

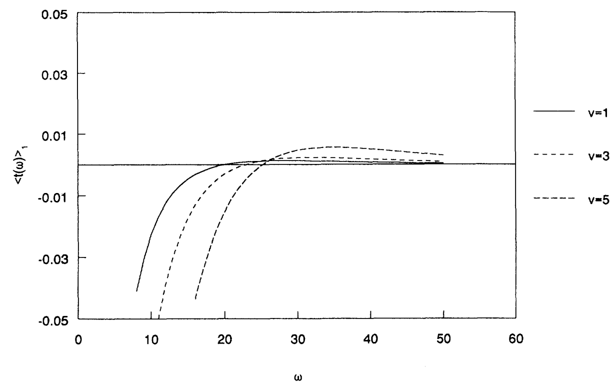

FIG. 1. Curves of $\langle t(\omega)\rangle_1$ plotted as a function of ω for $v=1$, 3, and 5. The resonant behavior increases with increasing v.

Here again the integrals are sufficiently simple so that they can be evaluated explicitly. If we write

$$s=\frac{v^2}{4}+i\omega , \quad (20)$$

then

$$\langle t(\omega)\rangle_1=\frac{e^v}{v}\text{Re}\left\{\frac{e^{-vs}}{i\omega}\right\}. \quad (21)$$

This can be further reduced to

$$\langle t(\omega)\rangle_1=-\frac{e^{v-a}\sin b}{v\omega} , \quad (22)$$

in which the parameters a and b are

$$a=\{\tfrac{1}{8}[v^2+(v^4+16\omega^2)^{1/2}]\}^{1/2} , \quad b=\frac{\omega}{2a} . \quad (23)$$

A plot of $\langle t(\omega)\rangle_1$ as a function of ω is shown in Fig. 1 from which the resonant behavior is evident, although a glance at the ordinate values shows that it is quite a weak one for the values of v that have been chosen for the graphs. The general mathematical techniques in the present paper also allow a calculation of higher moments. This has not been included here nor have we given the probability density for the FPT because it is expressed in terms of a somewhat complicated integral that can only be evaluated numerically. It is not known at present whether higher-order terms in the perturbation series also exhibit resonant behavior as a function of ω, although we believe it to be true.

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